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MEMORANDUM REPORT NO. 2705

FACTORIAL AND HADAMARD SERIES FOR BESSEL FUNCTIONS OF ORDERS ZERO AND ONE

Alexander S. Elder
Emma M. Wineholt

December 1976

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USA BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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Bessel functions of orders zero and one for moderate and large positive arguments have been programmed in FORTRAN using factorial series for $J_n(x)$, $Y_n(x)$ and $K_n(x)$ and Hadamard series for $I_n(x)$. A subroutine to calculate Stirling numbers of the first kind was developed for use in the factorial series. The recurrence relation was modified OVER →

JSUBN(X), YSUBN(X) AND KSUBN(X) ISUBN(X)

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and the resulting Stirling numbers scaled so that the entire range of the computer was utilized; e.g., $10^{-150} < S < 10^{150}$ instead of $10^{0} < S < 10^{150}$. In this way, more terms of the series can be calculated and higher accuracy obtained. For use in the Hadamard series, a subroutine to calculate incomplete gamma functions was developed. Various algorithms were necessary to encompass the required range of arguments.

These programs were devised to verify the accuracy (for moderate and large arguments) of our previously developed Bessel function subroutine. These programs replace the asymptotic series with convergent series, which, of course, is desirable. Extension of the program to complex arguments is now in progress.

10 TO THE MINUS 150th POWER < S < 10 TO THE 150th POWER

0 < S < 10 TO THE 150th POWER

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LIST OF SYMBOLS

a	parameter
j	index
k	index
m	index
n	order of Bessel function
p	index
r	index
x	argument
A_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
B_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
C_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
D_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
E_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
F	scale factor used in modified Stirling numbers
F_j	coefficients in an asymptotic series, $j=0,1,2,\ldots$
$G(x)$	factor representing the difference of two asymptotic series
$H(x)$	factor representing the sum of two asymptotic series
$I_n(x)$	modified Bessel function of the first kind of order n and argument x
$J_n(x)$	ordinary Bessel function of the first kind of order n and argument x
$K_n(x)$	modified Bessel function of the second kind of order n and argument x
$M(x)$	factor representing the sum of two asymptotic series
$M(a,b,x)$	Kummer function with arguments a, b, and x
$N(x)$	factor representing the difference of two asymptotic series
P_n	factor representing the sum of an asymptotic series
Q_n	factor representing the sum of an asymptotic series
S	modified Stirling number of the first kind
S_n	factor representing the sum of an asymptotic series

LIST OF SYMBOLS (CONT'D)

T_n	partial sum of S_n
$V_{n,r}$	numerator in each term of T_n
$W_{o,n}(x)$	Whittaker function for argument x and indices 0 and n
$Y_n(x)$	ordinary Bessel function of the second kind of order n and argument x
$\gamma(a,b)$	incomplete gamma function with arguments a and b
θ	integration variable
ν	index
Γ_k^ν	Stirling number of the first kind

I. INTRODUCTION

Factorial series derived from the Laplace integral converge rapidly for large values of the argument and, thus, are preferable to the corresponding asymptotic series. However, the traditional algorithm leads to very large numbers and must be modified if it is to be useful for numerical work. One procedure for scaling the large Stirling numbers which occur in the analysis is derived below.

Factorial series based on a Laplace integral evaluated between finite limits will generally diverge, so that an alternate procedure is required. One method, due to Hadamard, is to expand the Laplace integral in a series of incomplete gamma functions. The resulting series converge rapidly for large values of the argument. In practice, expansions in terms of the Kummer function are more convenient for computation. These functions are closely related to the incomplete gamma function.

Computer programs based on these algorithms will be used to check the accuracy of the BRL subroutines for Bessel functions of complex argument and integral order. This is necessary as tables are not available for a sufficient range of order and argument to make a detailed check by comparison.

II. FACTORIAL SERIES

The factorial series are used to calculate $K_n(x)$, $J_n(x)$ and $Y_n(x)$. $K_n(x)$ can be expressed in terms of the Whittaker function as[1]

$$K_n(x) = \left(\frac{\pi}{2x}\right)^{1/2} W_{o,n}(2x),$$

where the asymptotic expansion for the Whittaker function is[2]

$$W_{o,n}(2x) = e^{-x}\left\{1 + \sum_{m=1}^{\infty}\frac{[n^2-(-1/2)^2]\,[n^2-(-3/2)^2]\ldots[n^2-(1/2-m)^2]}{m!\,(2x)^m}\right\}$$

[1] *Handbook of Mathematical Functions, NBS55, U.S. Government Printing Office, 1964, p. 377.*

[2] *Modern Analysis, E. J. Whittaker and G. N. Watson, University Press, Cambridge, England, 1927, p. 343.*

This asymptotic expansion was derived from a Laplace integral evaluated between zero and infinity and involves only negative integral powers of the argument.

For n = 0,

$$K_o(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left\{1 - \frac{1^2}{1!(8x)} + \frac{1^2 \cdot 3^2}{2!(8x)^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \ldots\right\}$$

$$= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left\{\sum_{j=o}^{k} \frac{A_j}{x^j}\right\}$$

For n = 1,

$$K_1(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left\{1 + \frac{1 \cdot 3}{1!(8x)} - \frac{1^2 \cdot 3 \cdot 5}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{3!(8x)^3} - \ldots\right\}$$

$$= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left\{\sum_{j=o}^{k} \frac{B_j}{x^j}\right\}$$

A computer tabulation of the first fifty of these coefficients is shown in Table I.

These series can be summed by convergent factorial series using an algorithm described by Wasow:[3]

$$x^{-p} = \sum_{r=p-1}^{\infty} \frac{\Gamma^r_{r-p+1}}{x(x+1)(x+2)\ .\ .\ .\ (x+r)} \quad ,$$

where Γ denotes the Stirling numbers of the first kind.[3]

Now, $K_o(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} S_o$,

where $S_o = 1 - \frac{1^2}{1!(8x)} + T_o$

$$T_o = \sum_{j=2}^{k} \frac{A_j}{x^j} = \frac{A_2}{x^2} + \frac{A_3}{x^3} + \ldots$$

---

[3] *Asymptotic Expansions for Ordinary Differential Equations, W. Wasow, Interscience Publishers, John Wiley, NY, 1965, p. 330.*

J	A(J)	B(J)
0	0.100000000000000E 01	0.100000000000000E 01
1	-0.125000000000000E 00	0.375000000000000E 00
2	0.703125000000000E-01	-0.117187500000000E 00
3	-0.732421875000000E-01	0.102539062500000E 00
4	0.112152099609375E 00	-0.144195556640625E 00
5	-0.227108001708984E 00	0.277576446533203E 00
6	0.572501420974731E 00	-0.676592588424683E 00
7	-0.172772750258446E 01	0.199353173375130E 01
8	0.607404200127348E 01	-0.688391426810995E 01
9	-0.243805296995561E 02	0.272488273112685E 02
10	0.110017140269247E 03	-0.121597891876536E 03
11	-0.551335896122021E 03	0.603844076705070E 03
12	0.303809051092238E 04	-0.330227229448085E 04
13	-0.182577554742932E 05	0.197183759122366E 05
14	0.118838426256783E 06	-0.127641272646175E 06
15	-0.832859304016289E 06	0.890297876707068E 06
16	0.625295149343480E 07	-0.665636771881769E 07
17	-0.500695895319889E 08	0.531041101096852E 08
18	0.425939216504767E 09	-0.450278600305039E 09
19	-0.383625518023043E 10	0.404362032510775E 10
20	0.364684008070656E 11	-0.383385752074279E 11
21	-0.364901081884983E 12	0.382701134659861E 12
22	0.383353466139394E 13	-0.401183859913320E 13
23	-0.421897157028410E 14	0.440648141785228E 14
24	0.485401468685290E 15	-0.506056850331473E 15
25	-0.582724463156691E 16	0.606509135122270E 16
26	0.728685734937766E 17	-0.757261646111796E 17
27	-0.947628809926011E 18	0.983388387659068E 18
28	0.127972194197597E 20	-0.132625728532056E 20
29	-0.179216232305170E 21	0.185504521157983E 21
30	0.259938210272623E 22	-0.268749675027628E 22
31	-0.390012129203400E 23	0.402799412128102E 23
32	0.604671148753240E 24	-0.623867058237470E 24
33	-0.967702880106985E 25	0.997478353341046E 25
34	0.159706552529421E 27	-0.164473912306419E 27
35	-0.271558177354491E 28	0.279429428872012E 28
36	0.475321101404162E 29	-0.488710428204280E 29
37	-0.855738563980669E 30	0.879183456144523E 30
38	0.158339783631292E 32	-0.162562177861459E 32
39	-0.300896338830105E 33	0.308711828150368E 33
40	0.586841890824589E 34	-0.601698647554326E 34
41	-0.117386269685980E 36	0.120284696097979E 36
42	0.240676789246046E 37	-0.246476229950770E 37
43	-0.505491221599616E 38	0.517385132696078E 38
44	0.108694973189986E 40	-0.111193708205847E 40
45	-0.239159134066077E 41	0.244533496629359E 41
46	0.538173040543799E 42	-0.550001019456850E 42
47	-0.123794112437854E 44	0.126456351415012E 44
48	0.290948402279072E 45	-0.297073631800736E 45
49	-0.698350387000965E 46	0.712749364052532E 46

Table I. Coefficients for Asymptotic Series

Applying Wasow's algorithm to these terms, we obtained

$$\frac{A_2}{x^2} = A_2 \left(\frac{\Gamma_o^1}{x(x+1)} + \frac{\Gamma_1^2}{x(x+1)(x+2)} + \frac{\Gamma_2^3}{x(x+1)(x+2)(x+3)} + \cdots \right)$$

$$\frac{A_3}{x^3} = A_3 \left(\frac{\Gamma_o^2}{x(x+1)} + \frac{\Gamma_1^3}{x(x+1)(x+2)} + \frac{\Gamma_2^4}{x(x+1)(x+2)(x+3)} + \cdots \right)$$

Therefore, T_o can be expressed as

$$T_o = \sum_{r=1}^{\infty} \frac{V_{o,r}}{x(x+1)\ \cdots\ (x+r)} ,$$

where $V_{o,r} = A_2\, \Gamma_{r-1}^r + A_3\, \Gamma_{r-2}^r + A_4\, \Gamma_{r-3}^r + \cdots .$

These coefficients can be calculated and stored in the memory of the computer for recall on demand. The calculations for these coefficients, involving Stirling numbers, lead to very large numbers in the computation of high-order terms.

Since the Stirling numbers are always greater than or equal to one, we modified them for optimal use of the full range of the computer.

The Stirling numbers were modified in the following way:

$$S_k^\nu = F\, \Gamma_k^\nu/(\nu-1)!, \quad F = \text{scale factor, such as } 10^{125}$$

$$S_o^1 = F$$

$$S_o^\nu = S_o^{\nu-1}/(\nu-1)$$

$$S_k^\nu = S_{k-1}^{\nu-1} + S_k^{\nu-1}/(\nu-1)$$

The scale factor and the number of modified Stirling numbers which can be calculated are machine-dependent. The computers at BRL have a range from 10^{-155} to 10^{155}, single precision, which is larger than the range of most computers. As can be seen from Table II, for $F = 10^{125}$ and $n = 150$, the modified Stirling numbers range from 10^{-135} to 10^{125}. The process of scaling the Stirling numbers in this way must then be reversed in calculating each term of the factorial series.

By this transformation we obtained accurate results (15 significant digits) for $x \geq 6$ by summing 150 terms. Similar accuracy could be obtained on most computers using double precision.

1	0.262541431038901-135	0.293390049185972-131	0.162469629570885-127
4	0.594416307877133-124	0.161631807256184-120	0.348403288776085-117
7	0.620095188981095-114	0.937263899085794-111	0.122805989253782-107
10	0.141690995161720-104	0.145745824899596-101	0.134994681511499E-98
13	0.113521536148685E-95	0.872724632875019E-93	0.616964862448953E-90
16	0.403105100240183E-87	0.244485921578491E-84	0.138176751796042E-81
19	0.730187886089758E-79	0.361878183982837E-76	0.168651283284991E-73
22	0.740917060108629E-71	0.307506122625604E-68	0.120810725011220E-65
25	0.450102130711339E-63	0.159290231850353E-60	0.536288624778063E-58
28	0.172006626264944E-55	0.526244225186191E-53	0.153759178190215E-50
31	0.429520654391673E-48	0.114831115329136E-45	0.294089524864916E-43
34	0.722149952063180E-41	0.170161172544539E-38	0.385045773882694E-36
37	0.837326275585929E-34	0.175105193623330E-31	0.352369350644976E-29
40	0.682727806061237E-27	0.127434483546970E-24	0.229266634594032E-22
43	0.397758943581573E-20	0.665766457973381E-18	0.107555387883677E-15
46	0.167774015729906E-13	0.252791612712235E-11	0.368043389831136E-09
49	0.517937210655300E-07	0.704743584443506E-05	0.927441656806729E-03
52	0.118075771165955E 00	0.145466542291410E 02	0.173458453446450E 04
55	0.200240793190783E 06	0.223832004301461E 08	0.242317917476252E 10
58	0.254107442106383E 12	0.258158475955319E 14	0.254129580523208E 16
61	0.242426888397106E 18	0.224137435817219E 20	0.200863910466421E 22
64	0.174495356761975E 24	0.146958814419687E 26	0.119996196936958E 28
67	0.950002380342511E 29	0.729268853726516E 31	0.542840153293270E 33
70	0.391821615456175E 35	0.274247724384982E 37	0.186138969273117E 39
73	0.122508995716187E 41	0.781856783017240E 42	0.483840847139701E 44
76	0.290319945584933E 46	0.168899572458664E 48	0.952645306311109E 49
79	0.520899483162194E 51	0.276096523091475E 53	0.141844276858788E 55
82	0.706254632117985E 56	0.340768370204050E 58	0.159312784381071E 60
85	0.721564602846948E 61	0.316568545939851E 63	0.134510966747108E 65
88	0.553438118324812E 66	0.220455227036434E 68	0.850010949437138E 69
91	0.317167294679423E 71	0.114501970441310E 73	0.399846121274463E 74
94	0.135025685209439E 76	0.440823733512481E 77	0.139095522997561E 79
97	0.424060880287624E 80	0.124873374341855E 82	0.355050280369826E 83
100	0.974387656698037E 84	0.258006379428166E 86	0.658888038138901E 87
103	0.162215177246175E 89	0.384836375802625E 90	0.879348340649532E 91
106	0.193432814981343E 93	0.409407659018037E 94	0.833293416906864E 95
109	0.163005483906049E 97	0.306267211155246E 98	0.552343491325650E 99
112	0.955495084811848+100	0.158431163016062+102	0.251599210263367+103
115	0.382364608112042+104	0.555606274242301+105	0.771214486559342+106
118	0.102158281160150+108	0.129004865734295+109	0.155126469001113+110
121	0.177415994653956+111	0.192739054938949+112	0.198618366362429+113
124	0.193865171243942+114	0.178944908244716+115	0.155930607458550+116
127	0.128034645087299+117	0.988621557189148+117	0.716280594664574+118
130	0.485782753348748+119	0.307581787025027+120	0.181290891228498+121
133	0.991494859813093+121	0.501359095737511+122	0.233459123119021+123
136	0.996588263274595+123	0.388006214031004+124	0.136972391170442+125
139	0.435467363384560+125	0.123698885459199+126	0.311019186727229+126
142	0.684403248787668+126	0.129991588080456+127	0.209419799774947+127
145	0.279759712120229+127	0.300552651141317+127	0.248534593164330+127
148	0.147742753080902+127	0.558451392197723+126	0.100000000000000+126

Table II. Modified Stirling Numbers for n = 150

Similarly, $K_1(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} S_1$,

where $S_1 = 1 + \frac{1 \cdot 3}{1!(8x)} + T_1$

$$T_1 = \sum_{r=1}^{\infty} \frac{V_{1,r}}{x(x+1) \ldots (x+r)} ,$$

where $V_{1,r} = B_2\Gamma^r_{r-1} + B_3\Gamma^r_{r-2} + B_4\Gamma^r_{r-3} + \ldots$

The results for $K_1(x)$ were equally accurate.

The asymptotic series for the ordinary Bessel functions, $x \leq 25$, are:[4]

$$J_o(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_o(x) \cos(x - \frac{\pi}{4}) - Q_o(x) \sin(x - \frac{\pi}{4})]$$

$$J_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_1(x) \cos(x - \frac{3\pi}{4}) - Q_1(x) \sin(x - \frac{3\pi}{4})]$$

$$Y_o(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_o(x) \sin(x - \frac{\pi}{4}) + Q_o(x) \cos(x - \frac{\pi}{4})]$$

$$Y_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} [P_1(x) \sin(x - \frac{3\pi}{4}) + Q_1(x) \cos(x - \frac{3\pi}{4})] ,$$

where $P_o(x) \sim 1 - \frac{1^2 \cdot 3^2}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{4!(8x)^4} - \ldots$

$$= \sum_{j=o}^{k} \frac{C_j}{x^{2j}}$$

and $Q_o(x) \sim -\frac{1^2}{1!(8x)} + \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{5!(8x)^5} + \ldots$

$$= \sum_{j=o}^{k} \frac{D_j}{x^{2j+1}}$$

[4] *Bessel Functions, Part I, published by British Association for the Advancement of Science, University Press, Cambridge, England, 1937, p. 202.*

Note that $C_o = |A_o|,\ C_1 = -|A_2|,\ .\ .\ .\ .\ ,\ C_j = (-1)^j|A_{2j}|$

and $D_o = -|A_1|,\ D_1 = |A_3|,\ .\ .\ .\ .\ ,\ D_j = (-1)^{j+1}|A_{2j+1}|$

Similarly,

$$P_1 \sim \sum_{j=o}^{k} \frac{E_j}{x^{2j}} \quad \text{and} \quad Q_1 \sim \sum_{j=o}^{k} \frac{F_j}{x^{2j+1}}$$

And, again, $E_o = |B_o|,\ E_1 = -|B_2|,\ .\ .\ .\ .\ ,\ E_j = (-1)^j|B_{2j}|$

$$F_o = |B_1|,\ F_1 = -|B_3|,\ .\ .\ .\ .\ ,\ F_j = (-1)^j|B_{2j+1}|$$

For the ordinary Bessel functions, $x > 25$,

$$J_o(x) = G(x)\sin(x) + H(x)\cos(x)$$

$$J_1(x) = M(x)\sin(x) - N(x)\cos(x)$$

$$Y_o(x) = H(x)\sin(x) - G(x)\cos(x)$$

$$Y_1(x) = -N(x)\sin(x) - M(x)\cos(x)\ ,$$

where

$$G(x) = (\pi x)^{-1/2}\,[P_o(x)-Q_o(x)]$$

$$H(x) = (\pi x)^{-1/2}\,[P_o(x)+Q_o(x)]$$

$$M(x) = (\pi x)^{-1/2}\,[P_1(x)+Q_1(x)]$$

$$N(x) = (\pi x)^{-1/2}\,[P_1(x)-Q_1(x)]$$

So, for $x > 25$, the same coefficients are merely arranged in a different manner.

As before, the results obtained were accurate to 15 significant digits for $x \geq 6$ by summing 150 terms. A sample tabulation of the ordinary Bessel functions from the computer is shown in Table III.

We attempted to calculate $I_n(x)$ in the same manner, but the factorial series diverged.

III. HADAMARD SERIES

The factorial series for calculating $I_o(x)$ and $I_1(x)$ diverge since the Laplace integrals representing these functions are taken

X	K	J0	J1	Y0	Y1	
6	150	0.150645257250995E 00	-0.276683858127566E 00	-0.288194683981577E 00	-0.175010344300406E 00	FACT.SERIES
6		0.150645257250997E 00	-0.276683858127566E 00	-0.288194683981579E 00	-0.175010344300398E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
7	150	0.300079270519555E 00	-0.468282348234564E-02	-0.259497439672093E-01	-0.302667237024185E 00	FACT.SERIES
7		0.300079270519556E 00	-0.468282348234592E-02	-0.259497439672093E-01	-0.302667237024185E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
8	150	0.171650807137554E 00	0.234636346853915E 00	0.223521489387566E 00	-0.158060461731248E 00	FACT.SERIES
8		0.171650807137554E 00	0.234636346853915E 00	0.223521489387566E 00	-0.158060461731247E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
9	141	-0.903336111828761E-01	0.245311786573325E 00	0.249936698285025E 00	0.104314575196716E 00	FACT.SERIES
9		-0.903336111828762E-01	0.245311786573325E 00	0.249936698285025E 00	0.104314575196716E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
10	112	-0.245935764451348E 00	0.434727461688615E-01	0.556711672835995E-01	0.249015424206954E 00	FACT.SERIES
10		-0.245935764451349E 00	0.434727461688616E-01	0.556711672835995E-01	0.249015424206954E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
11	93	-0.171190300407196E 00	-0.176785298956722E 00	-0.168847323892079E 00	0.163705537414943E 00	FACT.SERIES
11		-0.171190300407196E 00	-0.176785298956722E 00	-0.168847323892079E 00	0.163705537414943E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
12	79	0.476893107968335E-01	-0.223447104490628E 00	-0.225237312634361E 00	-0.570992182608965E-01	FACT.SERIES
12		0.476893107968336E-01	-0.223447104490627E 00	-0.225237312634362E 00	-0.570992182608967E-01	SUBROUTINE
X	K	J0	J1	Y0	Y1	
13	69	0.206926102377068E 00	-0.703180521217785E-01	-0.782078645278760E-01	-0.210081408420693E 00	FACT.SERIES
13		0.206926102377068E 00	-0.703180521217787E-01	-0.782078645278759E-01	-0.210081408420693E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
14	61	0.171073476110459E 00	0.133375154698793E 00	0.127192568582184E 00	-0.166644841856172E 00	FACT.SERIES
14		0.171073476110458E 00	0.133375154698793E 00	0.127192568582184E 00	-0.166644841856172E 00	SUBROUTINE
X	K	J0	J1	Y0	Y1	
15	56	-0.142244728267808E-01	0.205104038613523E 00	0.205464296038918E 00	0.210736280368735E-01	FACT.SERIES
15		-0.142244728267808E-01	0.205104038613522E 00	0.205464296038919E 00	0.210736280368736E-01	SUBROUTINE

Table III. Computer Tabulation of Ordinary Bessel Functions

between finite limits and, therefore, cannot be expanded according to the previous algorithm. The Hadamard series, useful for large x, was used instead and has been programmed.

$I_n(x)$ can be expressed by:[5]

$$I_n(x) = \frac{(x/2)^n}{\Gamma(n+1/2)\Gamma(1/2)} \int_0^{\pi} e^{x\cos\theta} \sin^{2n}\theta \, d\theta$$

After expansion and term-by-term integration, the Hadamard series can then be written in the form

$$I_n(x) = \frac{e^x (2x)^{-1/2}}{\Gamma(n+1/2)\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{(1/2-n)_m \gamma(n+m+1/2, 2x)}{m!(2x)^m} ,$$

where γ denotes the incomplete gamma function and $(1/2-n)_m$ denotes Pochhammer's symbol.

$$(a)_n = a(a+1)(a+2) \ldots (a+n-1), \quad (a)_0 = 1$$

Each term in the expansion of these series contains the incomplete gamma function, which is expressed below in terms of the Kummer function.[6]

$$\gamma(a,x) = a^{-1} x^a e^{-x} M(1, 1+a, x) ,$$

where M denotes the Kummer function.

Hence, after substituting and simplifying, we have

$$I_n(x) = \frac{e^{-x}(2x)^n}{\Gamma(n+1/2)\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{(1/2-n)_m M(1, n+m+3/2, 2x)}{(n+m+1/2)\, m!} .$$

The solution of these series is straightforward and presented no problems in overflowing the memory of the computer. The calculation of the Kummer function required many terms (250 terms for x=75) to get the required accuracy. The solutions of the Hadamard series seem to have the correct convergent behavior. A sample computer tabulation is shown in Table IV.

---

[5] *Theory of Bessel Functions*, 2nd Ed., G. N. Watson, Macmillan Co., N.Y., 1948, p. 204.

[6] *Handbook of Mathematical Functions*, NBS 55, U.S. Government Printing Office, 1964, pp. 262, 504.

X	I0(X)	I1(X)	M	
17.0	0.235497022316829E 07	0.228462158380809E 07	25	HADAMARD SERIES
17.0	0.235497022316829E 07	0.228462158380808E 07		SUBROUTINE
18.0	0.621841242078100E 07	0.604313324211564E 07	22	HADAMARD SERIES
18.0	0.621841242078101E 07	0.604313324211563E 07		SUBROUTINE
19.0	0.164461904406117E 08	0.160073737858370E 08	21	HADAMARD SERIES
19.0	0.164461904406117E 08	0.160073737858370E 08		SUBROUTINE
20.0	0.435582825595536E 08	0.424549733851279E 08	19	HADAMARD SERIES
20.0	0.435582825595536E 08	0.424549733851278E 08		SUBROUTINE
21.0	0.115513961922158E 09	0.112729199137776E 09	18	HADAMARD SERIES
21.0	0.115513961922158E 09	0.112729199137776E 09		SUBROUTINE
22.0	0.306692993640365E 09	0.299639606877380E 09	17	HADAMARD SERIES
22.0	0.306692993640365E 09	0.299639606877379E 09		SUBROUTINE
23.0	0.815142122512894E 09	0.797220026089652E 09	17	HADAMARD SERIES
23.0	0.815142122512893E 09	0.797220026089651E 09		SUBROUTINE
24.0	0.216861908824138E 10	0.212294789328732E 10	16	HADAMARD SERIES
24.0	0.216861908824138E 10	0.212294789328732E 10		SUBROUTINE
25.0	0.577456060646632E 10	0.565786512987871E 10	15	HADAMARD SERIES
25.0	0.577456060646632E 10	0.565786512987871E 10		SUBROUTINE
26.0	0.153889767056608E 11	0.150900726423417E 11	15	HADAMARD SERIES
26.0	0.153889767056608E 11	0.150900726423417E 11		SUBROUTINE

Table IV. Computer Tabulation of $I_0(x)$ and $I_1(x)$

The results were good but not as accurate for moderate argument as we had hoped. We obtained 15 significant digits for $x \geq 17$ by summing 25 terms or less in the Hadamard series. This is not much better than the asymptotic series given by *

$$I_o(x) = \frac{e^x}{(2\pi x)^{1/2}} \left\{ 1 + \frac{1^2}{1!(8x)} + \frac{1^2 \cdot 3^2}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \ldots \right\}$$

$$I_1(x) = \frac{e^x}{(2\pi x)^{1/2}} \left\{ 1 - \frac{1 \cdot 3}{1!(8x)} - \frac{1^2 \cdot 3 \cdot 5}{2!(8x)^2} - \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{3!(8x)^3} - \ldots \right\}$$

When the asymptotic series were programmed, we obtained 15 significant digits for $x \geq 19$. However, the Hadamard series does provide an independent check on the accuracy of the asymptotic series used in our Bessel function subroutine.

IV. DISCUSSION AND CONCLUSIONS

Factorial series are an effective method for calculating modified Bessel functions of the second kind and related functions. Calculations have been limited to real arguments in this report; however, it is anticipated that extension of the algorithm to complex argument will not present any major difficulties.

On the other hand, the Hadamard series does not present any advantages over the usual asymptotic series, and, consequently, extending the algorithm to complex arguments is not planned. An expansion of the Laplace integrals for $I_o(x)$ and $I_1(x)$ in terms of the incomplete beta function is now being developed and should overcome difficulties encountered with the Hadamard series.

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* Reference 4, p. 271.

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